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ON THE PROBLEM OF SHEAR-LOCKING IN FINITE ELEMENTS BASED ON SHEAR DEFORMABLE PLATE THEORY

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Abstract—The phenomenon of shear locking in plate finite elements, or the loss of accuracy when thin plates are modelled by shear deformable elements, is explained in terms of the presence of boundary layer-type solutions to the equations of shear deformable plate theory, coupled with existing arguments in the literature. To demonstrate this, the governing equations of Reissner shear deformable theory were derived and reduced to two independent equations expressed in terms of a displacement potential ϕ and rotational stream function ψ for a transversely isotropic plate. A four node thirty-six degree of freedom C^2 continuous plate finite element which ignored the edge-effect equation in ψ, was derived using an interpolation of the displacement potential ϕ. A similar finite element based on classical plate theory was also shown. A square plate with simply-supported edges was modelled using these finite elements over a wide range of span-to-thickness ratios. All results converged rapidly to accepted solutions and did not exhibit shear-locking behavior under full integration. A discussion on the actual cause of shear-locking and recommendations for future development and implementation of the concepts in this study were made.

INTRODUCTION

The interest in shear deformable plate bending theory has increased significantly over the past twenty years primarily due to the increased use of laminated composite materials in structural applications. These materials display low transverse (out-of-plane) shear properties which must be accounted for in many analyses. For example, the accurate calculation of adherend displacements is essential when modelling adhesively bonded laminated composite joints. The finite element method has similarly become an invaluable tool to the analyst for determining the behavior of structures under many different loading configurations. As a result many displacement-based finite elements have been developed which are capable of modelling shear deformable behavior in plates. Unfortunately most of these elements become very stiff when used to model thin structures, resulting in solutions which are much smaller than the exact solution. This effect is termed shear-locking, and much effort has been aimed at identifying and eliminating the source of the problem. Reviews of shear deformable plate finite elements can be found in Zienkiewicz and Taylor (1991), Averill (1989), and Averill and Reddy (1990).

The most successful technique for alleviating the problems associated with shear-locking is through evaluating certain transverse shear coefficients of the element stiffness matrix using a lower order numerical integration rule than that which is required to evaluate the coefficients exactly, as discussed in Zienkiewicz and Taylor (1991). This technique is called reduced or selective integration, and in many cases, has been used on elements which shear-lock when exact integration is performed. An inexact integration scheme, however, results in a rank deficient element stiffness matrix, which in turn, generates additional zero strain deformation modes in a solution, other than the rigid body movements. These are termed zero-energy modes and must be suppressed through stabilization techniques. All displacement-based shear deformable plate elements of this sort fail on some occasions, either by shear-locking or singular behavior. For this reason many researchers have directed their attention toward developing new elements which do not exhibit locking behavior and do not have zero-energy modes. Aside from traditional displacement-based models many other formulations have been investigated. These include mixed, hybrid, assumed strain, penalty and discrete Kirchhoff methods (see Averill and Reddy (1990)). Despite the large number of new elements reported in the literature, only a few have encountered some success. An element has yet to be developed which has no ostensible defect and is both easy to implement and computationally efficient. A better explanation of the cause of these failures is clearly needed so that robust elements can be designed and universally put to use.

In this study the cause of shear-locking is examined by rederiving the equations of shear deformable plate theory (SDPT) and making finite elements based on these equations. The complete equations of Reissner SDPT are derived using linear elasticity theory equations and the assumption that the thickness of the plate remains constant during deformation. The equations are separated into two independent governing equations in terms of the displacement potential ϕ and the rotational stream function ψ for a transversely isotropic plate. A finite element based solely on the equations involving the displacement potential ϕ was derived. The equations which included the rotational stream function ψ are associated with the boundary layer solutions and were ignored in the derivation. A similar finite element based on classical theory was also shown with a correction for shear displacements. Simply-supported moderately thick to thin square plates were modelled using these elements to examine the shear-locking phenomenon.

THEORY OF REISSNER SHEAR DEFORMABLE PLATE-BENDING

The governing equations of SDPT are derived here directly from elasticity theory equations with a certain physical hypothesis in order to produce a consistent theory. In contrast, others such as Hencky (1947), Reissner (1944), Reissner (1945), Reissner (1985) and Reissner (1986), have based their theories on variational principles with initial approximations of the stress and displacement distributions through the thickness of the plate to obtain the same results. The following derivation has been taken in part from Vasiliev (1992) to emphasize the form that the governing equations take and show their relevance to the finite element results. Refer to Fig. 1 for plate definitions. The three-dimensional elasticity theory equations that are necessary for this analysis are: the equilibrium equations,

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0,\quad \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z}=0,\quad \frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0 \tag{1}$$

the Hookian constitutive relations for a transversely isotropic material in the $x-y$ plane,

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)-\frac{\nu'}{E'}\sigma_z,\quad \gamma_{xy}=\frac{\tau_{xy}}{G}$$

$$\varepsilon_y=\frac{1}{E}(\sigma_y-\nu\sigma_x)-\frac{\nu'}{E'}\sigma_z,\quad \gamma_{xz}=\frac{\tau_{xz}}{G'}$$

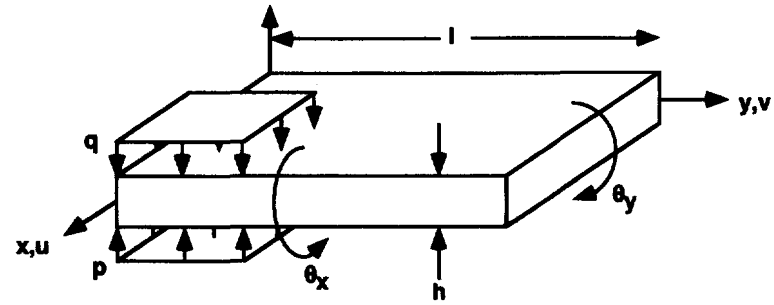

Fig. 1. The definitions of geometry, displacement and loading conditions for a square plate.

$$\varepsilon_z = \frac{1}{E'}[\sigma_z - \nu'(\sigma_x + \sigma_y)], \quad \gamma_{yz} = \frac{\tau_{yz}}{G'} \tag{2}$$

where $()'$ refers to properties in the out-of-plane z-direction, and the linear strain–displacement relations,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial y}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{3}$$

along with the following boundary conditions which must be satisfied on the surfaces of the plate,

$$\tau_{xz}(\pm h/2) = \tau_{yz}(h/2) = 0, \quad \sigma_z(-h/2) = -p, \quad \sigma_z(h/2) = -q. \tag{4}$$

Proposed originally by Thomson and Tait (1883), the only assumption which will be made in the present context of the theory is that the thickness of the plate remains constant during deformation. From eqn (3) this gives,

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0. \tag{5}$$

At this point it should be noted that the contribution of Mindlin (1951) to plate-bending theory is that the normal strain was not assumed to be zero as in (5) and in essence can be viewed as a refinement to SDPT.

In order to satisfy the elasticity eqns (1)–(3) with the constant thickness assumption (5) it follows that $E' \to \infty$ for arbitrary finite stresses (G' still remains finite however). Thus the introduction of a material which is flexible in transverse shear, and infinitely rigid in tension and compression in the direction of the z-axis is introduced to derive a well-conditioned two-dimensional theory. This operation was apparently first performed in plate theory by Kromm (1953). It follows from (5) that the deflection of the plate is $w = w(x, y)$ and the constitutive relations (2) become the plane-stress conditions,

$$\begin{gathered} \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}, \quad \gamma_{xz} = \frac{\tau_{xz}}{G'}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G'}. \end{gathered} \tag{6}$$

Since the material is assumed to be absolutely rigid in the z-direction from assumption (5), the actual distribution of shear stresses τ_{xz} and τ_{yz} through the thickness of the plate do not in themselves influence displacements. Only the resultants of τ_{xz} and τ_{yz} in the z-direction, given by the transverse forces,

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\,\mathrm{d}z, \quad Q_y = \int_{-h/2}^{h/2} \tau_{yz}\,\mathrm{d}z \tag{7}$$

are essential for the displacement field. Therefore in the context of a consistent plate theory based on assumption (5), the use of a shear-correction factor, which is often introduced to take into account the distribution of shear stresses through the thickness of the plate, is unfounded.

Transforming the integrals in (7) using the mean-value theorem gives,

$$Q_x = hs_x, \quad Q_y = hs_y \tag{8}$$

where average values s_x and s_y are the actual transverse shear stresses τ_{xz} and τ_{yz} at a point x, y, z_s.

The following relations are obtained from (6) along with the transverse shear strain–displacement relations in (3),

$$\frac{\partial u}{\partial z} = \tau_{xz}G' - \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial z} = \tau_{yz}G' - \frac{\partial w}{\partial y}. \tag{9}$$

Since only the resultants Q_x and Q_y influence the displacements of the plate, τ_{xz} and τ_{yz} in (9) can be replaced with s_x and s_y as given in (8). Expressions for the displacements are obtained by integrating (9) and using $w = w(x, y)$,

$$u = u^0 + z\theta_x, \quad v = v^0 + z\theta_y \tag{10}$$

where u^0 and v^0 are the displacements of points on the mid-plane surface $z = 0$ and where θ_x and θ_y are the rotations of the normal to that plane which takes the form,

$$\theta_x = \frac{Q_x}{C} - \frac{\partial w}{\partial x}, \quad \theta_y = \frac{Q_y}{C} - \frac{\partial w}{\partial y} \tag{11}$$

and where $C = G'h$ is the rigidity of the plate in transverse shear. Substituting displacements (10) into the strain–displacement relations (3) and considering only strains relevant to plate-bending (i.e., ignoring membrane effects) gives the geometric relations of the present theory,

$$\begin{gathered}\varepsilon_x = z\kappa_x, \quad \varepsilon_y = z\kappa_y, \quad \gamma_x = z\kappa_{xy} \\ \kappa_x = \frac{\partial \theta_x}{\partial x}, \quad \kappa_y = \frac{\partial \theta_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}.\end{gathered} \tag{12}$$

Using (6) and (12) the stresses become

$$\sigma_x = \frac{E}{1-\nu^2} z(\kappa_x + \nu\kappa_y), \quad \sigma_y = \frac{E}{1-\nu^2} z(\kappa_y + \nu\kappa_x), \quad \tau_{xy} = \frac{E}{2(1+\nu)} z\kappa_{xy}. \tag{13}$$

The physical relations of the theory are given by the moment resultants using (13),

$$\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x z \, dz = D(\kappa_x + \nu\kappa_y) \\ M_y &= \int_{-h/2}^{h/2} \sigma_y z \, dz = D(\kappa_y + \nu\kappa_x) \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} z \, dz = \frac{D(1-\nu)}{2} \kappa_{xy} \end{aligned} \tag{14}$$

where $D = [Eh^3/12(1-\nu^2)]$ is the flexural rigidity of the plate, and the transverse force resultants from (11),

$$Q_x = C\gamma_{xz}^0, \quad Q_y = C\gamma_{yz}^0 \tag{15}$$

where γ_{xz}^0 and γ_{yz}^0 are defined as average through-the-thickness transverse shear strains of the form,

$$\gamma_{xz}^0 = \theta_x + \frac{\partial w}{\partial x}, \quad \gamma_{yz}^0 = \theta_y + \frac{\partial w}{\partial y}. \tag{16}$$

Using (14) the deformations from (13) can be eliminated and given in terms of the moment resultants bringing them to their final form,

$$\sigma_x = \frac{12M_x}{h^3}z, \quad \sigma_y = \frac{12M_y}{h^3}z, \quad \tau_{xy} = \frac{12M_{xy}}{h^3}z. \tag{17}$$

In order to find a similar expression for τ_{xz} and τ_{yz}, eqns (17) are substituted into the first two equilibrium equations in (1) and integrated over the z-axis. Using the conditions $\tau_{xz}(-h/2) = \tau_{yz}(-h/2) = 0$ in (4) gives expressions for the transverse shear stresses,

$$\tau_{xz} = \frac{3}{2h}\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right)\left(1 - \frac{4z^2}{h^2}\right), \quad \tau_{yz} = \frac{3}{2h}\left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}\right)\left(1 - \frac{4z^2}{h^2}\right) \tag{18}$$

noting that the conditions $\tau_{xz}(h/2) = \tau_{yz}(h/2) = 0$ are satisfied due to symmetry. It is convenient to express τ_{xz} and τ_{yz} in terms of the their respective transverse force resultants by substituting (18) into (7) to give the equilibrium equations of an element of the plate,

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0. \tag{19}$$

With these equations, (18) becomes,

$$\tau_{xz} = \frac{3Q_x}{2h}\left(1 - \frac{4z^2}{h^2}\right), \quad \tau_{yz} = \frac{3Q_y}{2h}\left(1 - \frac{4z^2}{h^2}\right). \tag{20}$$

The coordinate z_s of the point at which the average stresses s_x and s_y operate can now be found from (8) and (20) which gives $z_s = \pm h/2(2\sqrt{3})$.

To determine the normal stress σ_z the last equilibrium equation in (1) is integrated over the z-axis. Using (20) and the second condition in (4) gives,

$$\sigma_z = -p - \left(\frac{1}{2} + \frac{3z}{2h} - \frac{2z^3}{h^3}\right)\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right). \tag{21}$$

Satisfying the last boundary condition in (4) the last plate-element equilibrium equation becomes,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \bar{p} = 0 \tag{22}$$

where $\bar{p} = p + q$. This brings (21) into the final form,

$$\sigma_z = -p+(p-q)\left(\frac{1}{2}+\frac{3z}{2h}-\frac{2z^3}{h^3}\right). \tag{23}$$

Thus the plate-bending theory discussed here reduces to (12), (14), (15), (19), and (22) which together give equations of the sixth-order.

Certain manipulations will now be made to the above plate-bending theory. With the physical relations (14) and (15), and the geometric relations (12), the equilibrium equations (19) and (22) can be written in terms of the generalized displacements w, θ_x and θ_y,

$$\begin{gathered} L_{11}(\theta_x)+L_{12}(\theta_y)+L_{13}(w) = 0 \\ L_{21}(\theta_x)+L_{22}(\theta_y)+L_{23}(w) = 0 \\ L_{31}(\theta_x)+L_{32}(\theta_y)+L_{33}(w)+\bar{p} = 0 \end{gathered} \tag{24}$$

where the differential operators L_{ij} are defined as

$$\begin{gathered} L_{11}() = \frac{D}{C}\frac{\partial^2()}{\partial x^2}+\frac{1}{k^2}\frac{\partial^2()}{\partial y^2}-(), \quad L_{12}() = L_2() = \frac{1}{k^2}\frac{1+v}{1-v}\frac{\partial^2()}{\partial x\,\partial y} \\ L_{13}() = -\frac{1}{C}L_{31}() = -\frac{\partial()}{\partial x}, \quad L_{22}() = \frac{D}{C}\frac{\partial^2()}{\partial y^2}+\frac{1}{k^2}\frac{\partial^2()}{\partial x^2}-() \\ L_{23}() = -\frac{1}{C}L_{32}() = -\frac{\partial()}{\partial y}, \quad L_{33}() = C\Delta() \\ k^2 = \frac{2C}{D(1-v)}, \quad \Delta() = \frac{\partial^2()}{\partial x^2}+\frac{\partial^2()}{\partial y^2}. \end{gathered} \tag{25}$$

This form of the plate-bending theory equations were first derived by Hencky (1947) using the Lagrange variational principle. Using an operator procedure the generalized displacements w, θ_x and θ_y can be written in terms of some function $F(x,y)$,

$$\begin{gathered} \theta_x = (L_{12}L_{23}-L_{13}L_{22})(F) = -\frac{\partial}{\partial x}L(F) \\ \theta_y = (L_{21}L_{13}-L_{23}L_{11})(F) = -\frac{\partial}{\partial y}L(F) \\ w = (L_{11}L_{22}-L_{12}L_{21})(F) = L(F)-\frac{D}{C}\Delta L(F) \\ L(F) = F-\frac{1}{k^2}\Delta F. \end{gathered} \tag{26}$$

It can be shown that the first three equations in (26) identically satisfy the first two equations in (24). From this procedure the last equation in (24) can be brought to the form,

$$D\Delta\Delta L(F) = \bar{p}. \tag{27}$$

Assuming that $L(F) \neq 0$ it can be shown that (27) is essentially of fourth-order.
Introducing the function

$$\phi = L(F) \tag{28}$$

eqns (26) and (27) can be written in the form

$$\theta_x = -\frac{\partial\phi}{\partial x}, \quad \theta_y = -\frac{\partial\phi}{\partial y}, \quad w = \phi - \frac{D}{C}\Delta\phi D\,\Delta\Delta\phi = \bar{p}. \tag{29}$$

As h becomes small $D/C \to 0$ and from (29) $\phi = w$ and the governing differential equation becomes that of classical plate theory. The fourth-order system (29) was derived in its essentials by Donnell (1959) and Donnell (1976). From these equations Donnell extended the use of transverse shear correction in beams proposed by Timoshenko (1921) to plates. In this method the transversal deflection of the plate can be represented by the sum of the deflection from classical theory w_{cl} and a correction for shear from the solution of the classical equations such that,

$$w = w_{cl} - \frac{D}{C}\Delta w_{cl}. \tag{30}$$

Equation (30) is an approximate solution in general since it is based on the condition that $\phi = w_{cl}$, and although ϕ and w_{cl} are described by the same biharmonic equation, each must generally satisfy different boundary conditions.

Since the sixth-order system (24) was reduced to the fourth-order eqn (29) by the substitution of (26) and (28), and recognizing that (29) is invalid for $L(F) = 0$ one more second-order equation must exist to complete the theory. As well it follows from (10) and the first two equations of (28) that the function ϕ is the potential of the displacement field in a plane z = const. Therefore from continuum mechanics the rotational plane motion can be described by the introduction of a stream function ψ which satisfies

$$\theta_x = \frac{\partial\psi}{\partial y}, \quad \theta_y = \frac{\partial\psi}{\partial x}, \quad w = 0. \tag{31}$$

Substituting these expressions into the last equation of (24) solves the homogeneous differential equation in (29). It also follows from (29) that

$$\frac{\partial}{\partial x}L(\psi) = 0, \quad \frac{\partial}{\partial y}L(\psi) = 0 \tag{32}$$

or $L(\psi) = \psi_0$. From (31) it can be seen that the constant ψ_0 does not affect the rotation field and can be assumed to be zero. Thus the final second-order equation becomes

$$\Delta\psi - k^2\psi = 0. \tag{33}$$

This equation was derived in various forms by a number of authors [e.g., Hencky (1947), Reissner (1944), Thomson and Tait (1883), Kromm (1953), and Donnell (1976)]. Equation (33) determines a state of stress that decays rapidly with increasing distance from the edge and is thus termed the edge-effect equation. The solution to this equation for the stream function ψ is the so-called 'boundary layer solution'.

Thus plate-bending theory reduces to the sixth-order system,

$$D\,\Delta\Delta\phi = \bar{p}, \quad \Delta\psi - k^2\psi = 0 \tag{34}$$

with the rotation angles and deflection given by

$$\theta_x = -\frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad \theta_y = -\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}, \quad w = \phi - \frac{D}{C}\Delta\phi. \tag{35}$$

The system (34) and (35) is known as the Reissner shear deformable plate theory. It can be shown that the transverse force and moment resultants take the form

$$M_x = -D\left[\frac{\partial^2\phi}{\partial x^2} + v\frac{\partial^2\phi}{\partial y^2} - (1-v)\frac{\partial^2\psi}{\partial x\,\partial y}\right]$$

$$M_y = -D\left[\frac{\partial^2\phi}{\partial y^2} + v\frac{\partial^2\phi}{\partial x^2} + (1-v)\frac{\partial^2\psi}{\partial x\,\partial y}\right]$$

$$M_{xy} = -D(1-v)\left[\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)\right]$$

$$Q_x = C\left(\theta_x + \frac{\partial w}{\partial x}\right) = -D\frac{\partial}{\partial x}\Delta\phi + C\frac{\partial\psi}{\partial y}$$

$$Q_y = C\left(\theta_y + \frac{\partial w}{\partial y}\right) = -D\frac{\partial}{\partial y}\Delta\phi - C\frac{\partial\psi}{\partial x}. \tag{36}$$

Some important remarks about the edge-effect equation in (33) should be made at this point. As previously mentioned, the solution to this equation is the boundary layer solution which decays rapidly with increasing distance from the edge of the plate. As an example, consider the plate in Fig. 1 and assume that it is infinitely long in the x-direction, and loaded in such a way that the stress–strain state does not depend on x. Equation (33) then becomes

$$\frac{d^2\psi}{d\bar{y}^2} - k^2\psi = 0 \tag{37}$$

where $\bar{y} = y/l$ and

$$k = 2\sqrt{3}\frac{l}{h} \tag{38}$$

for an isotropic plate. The solution to (37) is of the form,

$$\psi = C_1\,e^{-k\bar{y}} - C_2\,e^{k\bar{y}} \tag{39}$$

where $0 \leqslant \bar{y} \leqslant 1$ and C_1, C_2 are constants of integration that should be found from the boundary conditions at $\bar{y} = 0$ and $\bar{y} = 1$. For a plate with $l/h = 100$ (well within the range of thin plate behavior), the first exponent in the solution (39) is of the order 10^{-150}. Traditional numerical techniques identify this number with zero and will thus fail to find the constants C_1 and C_2. This is the mathematical mechanism which causes numerical methods, including the finite element method, to lose accuracy when dealing with problems that entirely involve boundary layer solutions.

It has been well documented that the physical mechanism of shear-locking in finite element analysis of plates is associated with the elastic energy of transverse shear strains in the functional of the total potential energy [e.g., Zienkiewicz and Taylor (1991)]. In order to have a finite element which does not shear-lock the formulation must be able to handle the constraint of zero shear strain in the limit that the thickness of the plate goes to zero. Using the notation of this study these constraints take the form,

$$\gamma^0_{xz} = \theta_x + \frac{\partial w}{\partial x} \to 0, \quad \gamma^0_{yz} = \theta_y + \frac{\partial w}{\partial y} \to 0 \tag{40}$$

as $h \to 0$. Shear locking results when these conditions are not satisfied. Most traditional displacement-based finite elements violate the conditions in (40) when fully integrated. Clearly these elements do not contain an explicit form of ϕ and ψ in their displacement and rotation fields. Using the expressions in (15) and (36), the transverse shear strains in the present context of SDPT take the form,

$$\gamma_{xz}^0 = -\frac{D}{C}\frac{\partial}{\partial x}\Delta\phi + \frac{\partial\psi}{\partial y}, \quad \gamma_{yz}^0 = -\frac{D}{C}\frac{\partial}{\partial y}\Delta\phi - \frac{\partial\psi}{\partial x}. \tag{41}$$

Here the first term in both equations are proportional to h^2 and clearly goes to zero when h becomes small for thin plates. The second terms however do not vanish in the thin plate limit. Note that these second terms are derivatives of the stream function ψ. Now it can be seen from eqns (34) and (35) that the shear deformation in SDPT demonstrates itself in two ways. First, it follows from the last equation in (35) that SDPT influences the plate deflections. Secondly, it induces the boundary layer solutions described by the second equation in (34). Calculations by Vasiliev (1992) have shown that the second effect is important only for special problems (e.g., pure torsion or contact problems). In most cases only the first effect, the influence of shear deformation on displacements, needs to be taken into account in calculations.

With this in mind it seems reasonable to ignore the boundary layer solution and derive a plane finite element which is based on this approximation. The transverse shear strains are now of the form

$$\gamma_{xz}^0 = -\frac{D}{C}\frac{\partial}{\partial x}\Delta\phi, \quad \gamma_{yz}^0 = -\frac{D}{C}\frac{\partial}{\partial y}\Delta\phi \tag{42}$$

which go to zero in the thin plate limit. An analogous transformation should be made to eqns (35) which reduce to

$$\theta_x = -\frac{\partial\phi}{\partial x}, \quad \theta_y = -\frac{\partial\phi}{\partial y}, \quad w = \phi - \frac{D}{C}\Delta\phi. \tag{43}$$

It should be emphasized that eqns (42) and (43) are approximate since the boundary layer solutions were ignored. There exists however at least one boundary problem in plate theory for which these equations are exact.

Consider a simply supported plate such that the conditions at the edges x = const are $w = 0$, $\theta_y = 0$, and $M_x = 0$ and at edges y = const are $w = 0$, $\theta_x = 0$ and $M_y = 0$. Using eqns (35) and (36), these conditions can be transformed to

$$\psi = 0, \quad \Delta\psi = 0, \quad \frac{\partial\psi}{\partial x} = 0 \quad \text{and} \quad \psi = 0, \quad \Delta\psi = 0, \quad \frac{\partial}{\partial y} = 0, \tag{44}$$

respectively. Thus the function $\psi(x, y)$ should satisfy the homogeneous eqn (33) and boundary conditions (44). This boundary value problem has only the trivial solution $\psi(x, y) \equiv 0$ and thus eqns (42) and (43) are exact for a simply supported plate.

PLATE BENDING FINITE ELEMENTS

Development of displacement based shear deformable plate-bending finite elements have traditionally been based on interpolations of geometric variables such as displacements and rotations. In doing so these finite elements cannot be explicitly separated into the two effects of shear deformation as previously mentioned. In this study a SDPT finite element was formulated which ignored the existence of the boundary layer solutions. In this derivation the fundamental variable to be interpolated is ϕ as defined in the previous section. A classical finite element formulation is also shown here to compare results with the SDPT element.

SDPT element

The strain energy of a plate element governed by (29) is given by,

$$U = \frac{1}{2}\iint \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\varepsilon}\,\mathrm{d}x\,\mathrm{d}y \tag{45}$$

where,

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz}^0 \\ \gamma_{yz}^0 \end{Bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} D & \nu D & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 \\ 0 & 0 & \dfrac{D(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & C \end{bmatrix}. \tag{46}$$

Substitution of the geometric relations (12), and the definitions of shear strains and rotations in (42) and (43) (which ignore the boundary layer solution) into the generalized strain matrix gives the strain energy in terms of ϕ where

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} -\dfrac{\partial^2\phi}{\partial x^2} \\ -\dfrac{\partial^2\phi}{\partial y^2} \\ -2\dfrac{\partial^2\phi}{\partial x\,\partial y} \\ -\dfrac{D}{C}\left[\dfrac{\partial^3\phi}{\partial x^3} + \dfrac{\partial^3\phi}{\partial x\,\partial y^2}\right] \\ -\dfrac{D}{C}\left[\dfrac{\partial^3\phi}{\partial x^2\,\partial y} + \dfrac{\partial^3\phi}{\partial y^3}\right] \end{Bmatrix} \tag{47}$$

which contains up to third derivatives of ϕ. In order to satisfy inter-element compatibility the interpolation functions for ϕ must be at least C^2 continuous. In other words deflection and rotations can only be continuous across element boundaries when up to second derivatives of ϕ are continuous. An interpolation which satisfies these requirements is the rectangular four node 36 degree-of-freedom element of Bogner *et al.* (1965). Their derivation is modified here to satisfy eqns (43). An orthogonal natural element coordinate system $\xi-\eta$ is chosen for this derivation such that ξ and η are collinear with the x and y directions respectively as shown in Fig. 2. The element geometry is modelled by linear Lagrangian shape functions of the form

$$x = \sum_{i=1}^{4} N_i^* x_{e_i} \quad \text{and} \quad y = \sum_{i=1}^{4} N_i^* y_{e_i}$$

where x_e and y_e are the location of the element corners in global coordinates and the shape functions N_i^* are given by

$$N_1^* = \tfrac{1}{4}(1-\xi)(1-\eta)$$

$$N_2^* = \tfrac{1}{4}(1+\xi)(1-\eta)$$

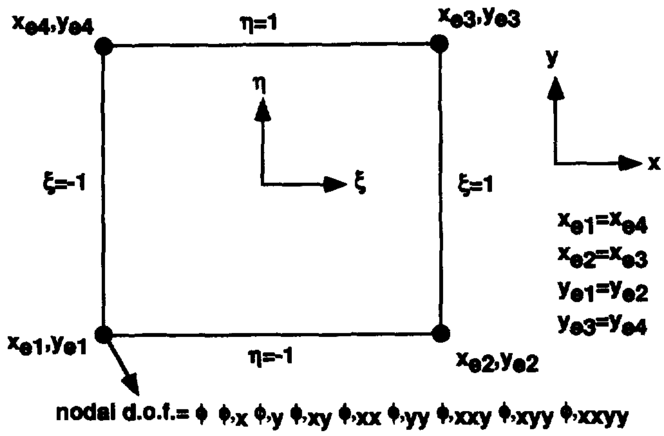

Fig. 2. The geometry and natural coordinate system of the 4-node 36 degree-of-freedom plate element.

$$N_3^* = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4^* = \tfrac{1}{4}(1-\xi)(1+\eta).$$

Transformation of derivatives from natural to global coordinates are performed using the Jacobian elements $J_{11} = \partial x/\partial\xi$ and $J_{22} = \partial y/\partial\eta$ such that

$$\frac{\partial}{\partial x} = \frac{1}{J_{11}}\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial y} = \frac{1}{J_{22}}\frac{\partial}{\partial\eta},\quad \frac{\partial^2}{\partial x\,\partial y} = \frac{1}{J_{11}J_{22}}\frac{\partial^2}{\partial\xi\,\partial\eta}\frac{\partial^2}{\partial\eta\partial x^2} = \frac{1}{J_{11}^2}\frac{\partial^2}{\partial\xi^2},\quad \frac{\partial^2}{\partial y^2} = \frac{1}{J_{22}^2}\frac{\partial^2}{\partial\eta^2}$$

$$\frac{\partial^3}{\partial x^3} = \frac{1}{J_{11}^3}\frac{\partial^3}{\partial\xi^3},\quad \frac{\partial^3}{\partial y^3} = \frac{1}{J_{22}^3}\frac{\partial^3}{\partial\eta^3}\frac{\partial^3}{\partial x^2\,\partial y} = \frac{1}{J_{11}^2J_{22}}\frac{\partial^3}{\partial\xi^2\,\partial\eta},\quad \frac{\partial^3}{\partial x\,\partial y^2} = \frac{1}{J_1 J_{22}^2}\frac{\partial^3}{\partial\xi\,\partial\eta^2}$$

$$\frac{\partial^4}{\partial x^2\,\partial y^2} = \frac{1}{J_{11}^2J_{22}^2}\frac{\partial^4}{\partial\xi^2\,\partial\eta^2}.$$

Complete quintic polynomial trial functions in the variables ξ and η are assumed for the interpolation of ϕ. The nodal degrees-of-freedom,

$$\phi \quad \phi_{,\xi} \quad \phi_{,\eta} \quad \phi_{,\xi\eta} \quad \phi_{,\xi\xi} \quad \phi_{,\eta\eta} \quad \phi_{,\xi\xi\eta} \quad \phi_{,\xi\eta\eta} \quad \phi_{,\xi\xi\eta\eta}$$

expressed in terms of derivatives in natural coordinated are located at element corners such that

$$\phi(\xi,\eta) = \sum_{i=1}^{36} N_i\delta_i' \tag{48}$$

where N_i are the interpolation function and δ_i' are the degrees-of-freedom. The generalized strains (47) are thus given in terms of the nodal degrees-of-freedom δ_i' according to,

$$\boldsymbol{\varepsilon} = \mathbf{B}\boldsymbol{\delta}'$$

where **B** is a five-by-thirty-six matrix containing the Jacobian elements and derivatives of the interpolation functions. The degrees-of-freedom δ_i' are converted into nodal degrees-of-freedom,

$$\phi \quad \phi_{,x} \quad \phi_{,y} \quad \phi_{,xy} \quad \phi_{,xx} \quad \phi_{,yy} \quad \phi_{,xxy} \quad \phi_{,xyy} \quad \phi_{,xxyy}$$

expressed in global coordinates by the transformation

$$\boldsymbol{\delta} = \mathbf{T}\boldsymbol{\delta}'$$

where **T** is a diagonal transformation matrix of the form

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 \\ 0 & J_{11} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & J_{11}^2 J_{22}^2 \end{bmatrix}.$$

The stiffness matrix is therefore given by

$$\mathbf{k} = \int_{-1}^{1} \int_{-1}^{1} |\mathbf{J}| \mathbf{T}\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{T} \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{49}$$

where $|\mathbf{J}| = J_{11}J_{22}$. The integrals in (49) are calculated exactly using a six-point Gaussian quadrature rule. The stiffness system for the plate element is therefore given by

$$\mathbf{P} = \mathbf{k}\boldsymbol{\delta}$$

where **P** is the generalized load vector. For the case of a uniformly distributed load $\bar{p}$, the load vector is derived from the work performed on the plate and is given by

$$\mathbf{P} = \int_{-1}^{1} \int_{-1}^{1} |\mathbf{J}| \mathbf{w}^{\mathrm{T}} \bar{p} \, \mathrm{d}\xi \, \mathrm{d}\eta \tag{50}$$

where the components of the two vector **w** are

$$w_i = \left(N_i - \frac{D}{C} \left[\frac{1}{J_{11}^2} \frac{\partial^2 N_i}{\partial \xi^2} + \frac{1}{J_{22}^2} \frac{\partial^2 N_i}{\partial \eta^2} \right] \right) T_{ii}. \tag{51}$$

The integrals in (50) are integrated exactly by at least a three-point Gaussian quadrature rule.

Assembly and the solution of multiple elements is performed in the usual manner [e.g., Cook *et al.* (1989)]. Upon solving the assembled system, transverse displacements in an element can be calculated at a point (ξ, η) from

$$w = (\xi, \eta) = \mathbf{w}(\xi, \eta)\boldsymbol{\delta}$$

where **w** is given by (51).

Classical plate element

A finite element based solely on classical plate theory was derived in a similar fashion as the shear deformable finite element above. This formulation is identical to the classical element of Bogner *et al.* (1965) except that a different natural element coordinate system was used. For this classical case $D/C \to 0$ and ϕ is replaced by w_{cl} everywhere along with the changes

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_{cl}}{\partial x} \\ -\dfrac{\partial^2 w_{cl}}{\partial y^2} \\ -2\dfrac{\partial^2 w_{cl}}{\partial x\, \partial y} \end{Bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & \dfrac{D(1-\nu)}{2} \end{bmatrix}$$

in (46). Although it can be seen that the classical element only requires C^1 continuity the higher order C^2 continuous interpolations in (48) were used to directly compare results with the shear deformable element solutions using (30). The components of the row vector $\mathbf{w}$ in (51) are now given by

$$w_i = N_i T_{ii}$$

and the classical transversal displacements are given by (52).

The classical plate element can also be used to give an approximate evaluation of the shear deformation effect by substituting the solution for w_{cl} obtained using this element into the expression for w in (30).

RESULTS

As a numerical example, consider a uniformly loaded simply-supported isotropic square plate. This configuration is convenient for two reasons. First, the boundary layer effect does not occur. Thus the results obtained from the SDPT element, which ignores the boundary layer solution, can be directly compared with the exact solution of the problem. It should be emphasized that the boundary layer effect does not occur only for a properly formulated boundary problem (i.e., for a rectangular domain with boundary conditions (44)). In traditional displacement-based plate finite elements the boundary conditions do not explicitly or implicitly coincide with (44). The boundary layer therefore does occur with these elements, and since polynomial interpolation functions cannot properly model this effect, the inconsistency of the boundary problem results in shear-locking behavior.

The second reason to use this simply-supported plate configuration is that an exact solution can be found for both the classical and shear deformable plate theories using the Navier method in Timoshenko and Woinowsky-Krieger (1959). Consider a rectangular domain $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, and present functions $\phi(x, y)$ and $\psi(x, y)$ in the form of a double Fourier series

$$\phi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn} \sin \lambda_m x \sin \lambda_n y$$

$$\psi(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \psi_{mn} \cos \lambda_m x \cos \lambda_n y \tag{53}$$

where $\lambda_m = m\pi/a$, $\lambda_n = n\pi/b$, and where ϕ_{mn} and ψ_{mn} are constant coefficients. Clearly each term of the second series in (53) satisfies boundary conditions (44). A uniform pressure p_0 decomposed in a similar series takes the form,

$$\bar{p}(x, y) = \frac{16p_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \sin \lambda_m x \sin \lambda_n y. \tag{54}$$

Substituting series (53) and (54) into eqns (34) and solving for the unknown coefficients ϕ_{mn} and ψ_{mn} gives the exact solution to the problem where

$$\phi_{mn} = \frac{16p_0}{\pi^6 D} \frac{1}{mn[(m/a)^2+(n/b)^2]^2}, \quad \psi_{mn} = 0. \tag{55}$$

Indeed this shows that $\psi(x, y) \equiv 0$. The plate deflections are found by substituting (53) and (55) into the last equation in (35) to give

$$w(x,y) = \frac{16p_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1+\dfrac{\pi^2 D}{C}[(m/a)^2+(n/b)^2]}{mn[(m/a)^2+(n/b)^2]^2} \sin\lambda_m x \sin\lambda_n y. \tag{56}$$

The Navier solution for classical plate theory is obtained by setting $C \to \infty$ in (56).

The conditions for a simply-supported edge x = constant when neglecting the edge-effect equations are given by

$$\begin{aligned} w &= \phi - \frac{D}{C}\Delta\phi = 0 \\ \phi_y &= -\frac{\partial\phi}{\partial y} = 0 \\ M_x &= -D\left[\frac{\partial^2\phi}{\partial x^2} + v\frac{\partial^2\phi}{\partial y^2}\right] = 0 \end{aligned} \tag{57}$$

which are all identically satisfied when

$$\phi = \frac{\partial\phi}{\partial y} = \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial y^2} = 0 \tag{58}$$

along that edge. Similarly the conditions for the classical theory are

$$\begin{aligned} w_{cl} &= 0 \\ \theta_y &= -\frac{\partial w_{cl}}{\partial y} = 0 \\ M_x &= -D\left[\frac{\partial^2 w_{cl}}{\partial x^2} + v\frac{\partial^2 w_{cl}}{\partial y^2}\right] = 0. \end{aligned} \tag{59}$$

Similar results are obtained for the edge y = constant.

A plate with these simply-supported boundary conditions on all edges was modelled by the shear deformable and classical theory elements previously described. Due to symmetry only one-quarter of the plate was modelled using a single element. The nodal fixity conditions for both cases are identical and are given in Fig. 3. It can be seen that the enforcement of both the geometric boundary conditions (i.e., edge displacement and tangential rotation) and the force boundary conditions (i.e., edge normal moments) can be performed simultaneously using (58). The non-dimensionalized center deflections of a plate $\bar{w} = wD/\bar{p}l^4$ with $v = 0.3$ over a wide range of span-to-thickness ratios were calculated. All numerical integrations were performed using a six-point Gaussian quadrature rule in double precision.

The center transversal deflection solutions for various span-to-thickness ratios are plotted in Fig. 4. It can clearly be seen that the use of one shear deformable element to model one-quarter of the plate gives excellent converged results and do not exhibit the phenomenon of shear-locking. The error in the finite element solution is compared with the exact solution given by (56) is less than 0.01% for the shear deformable element for all

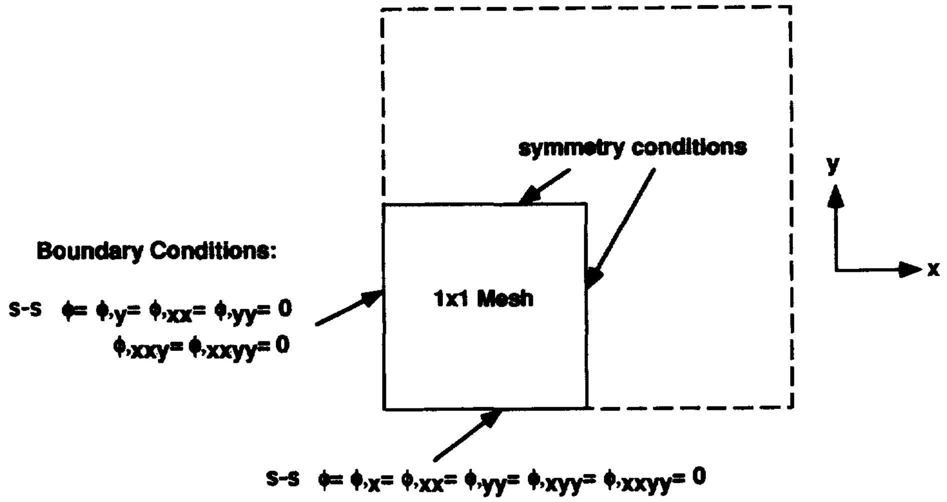

Fig. 3. The finite element model of a square plate with simply supported boundary conditions.

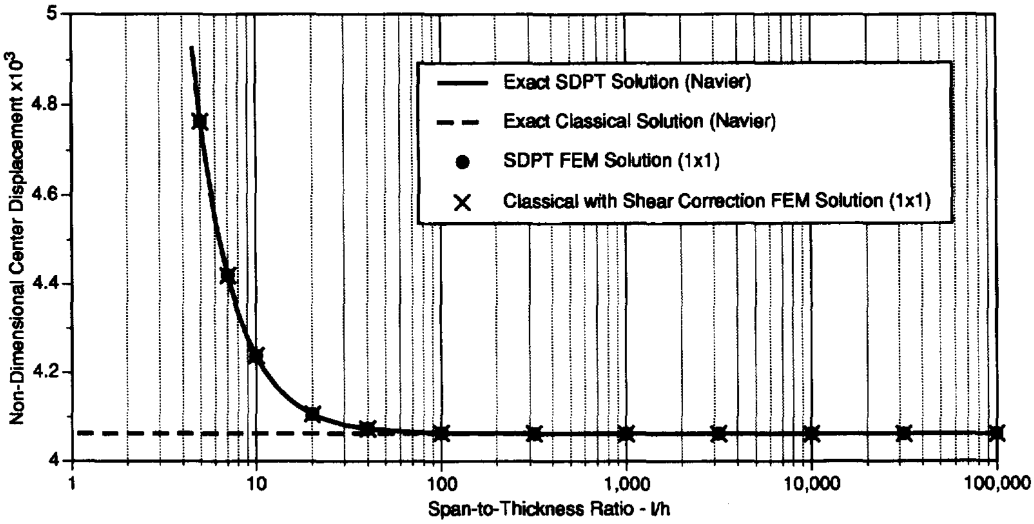

Fig. 4. The non-dimensionalized center deflections $\bar{w} = wD/\bar{p}l^4$ of a uniformly loaded, simply supported square plate with $\nu = 0.3$ modelled by the SDPT and classical plate finite elements over various span-to-thickness ratios.

span-to-thickness ratios. Refining the density of the mesh gave identical results. Preliminary results for clamped plates as well show that the SDPT finite element which ignores the boundary layer effect, does not shear lock.

Solutions based on the classical finite element with shear correction according to (30) practically coincide with the SDPT solutions, as shown in Fig. 4. This seems quite reasonable since boundary conditions (57) and (59) can be reduced to the same form $\phi = \partial^2\phi/\partial x^2 = 0$ and $w_{cl} = \partial^2 w_{cl}/\partial x^2 = 0$, respectively, thus making eqn (30) exact for the simply-supported plate. Although this method is only approximate for clamped plates, preliminary results suggest that these solutions are close to the SDPT finite element solutions.

DISCUSSION

Neglecting the existence of the edge-effect equation within the context of SDPT permitted the derivation of a plate finite element in ϕ which did not shear lock. The next logical step would be to incorporate the edge-effect equations into the finite element by interpolating with respect to ψ in some way. Since the two differential equations in (34) are independent, a superposition of two finite elements, one with respect to ϕ as before and the other in ψ would be possible. The two elements would be connected in the boundary conditions from the definitions of displacement and rotations. In this case all governing equations would be included and the results should converge to exact solutions. To incorporate these equations with a numerical technique, the boundary layer effect should be singled out and described analytically with the aid of solutions similar to (39).

From the work of Donnell the equations without the boundary layer solution were equivalent to solving the classical theory problem and using a shear correction on the displacements as in eqn (30). The solutions obtained by this second method converged for all test cases in this study and thus holds much promise for future implementation to plate problems.

The use of the above elements, which are C^2 continuous, poses an even greater challenge than the already problematic C_1 continuous plate elements when extending the solutions to non-rectangular plate geometries. The use of these elements are currently limited to rectangular plate applications only.

Preliminary work involving the 16 degree-of-freedom C^1 continuous interpolation of Bogner *et al.* (1965) to model the same system (29) shows promising results which do not shear lock. This may suggest that a lower order continuity element as is found in many finite element libraries may be sufficient to model plate displacements with shear correction at the expense of slower convergence rates. Further work in this area is needed to make the concepts in this study attractive for general finite element programs.

As mentioned previously, the separation of the governing relations for SDPT into the two independent differential equations for the fundamental unknowns ϕ and ψ can be done only for transversely isotropic material properties. For orthotropic plates, however, the problem is much more complicated and the exact separation of the governing relations into independent quantities is not possible. The equations do, however, have two solutions, a rapidly varying solution that corresponds to the boundary layer effect and a slowly varying solution that determines the ground state of the plate as in the transversely isotropic case. This coupling makes it nearly impossible to derive a plate finite element which neglects the edge-effect completely. However, through simple asymptotic considerations, the separation of the two solutions can be done approximately, as in Vasiliev (1992), and an element can therefore be constructed in a similar fashion as proposed for the transversely isotropic case.

CONCLUSIONS

The governing equations of Reissner shear deformation theory were derived from linear elasticity equations and the assumption that the thickness of the plate remains constant during deformation. The equations were separated into two independent governing equations expressed in terms of the displacement potential ϕ and the rotational stream function ψ for a transversely isotropic plate. The fourth-order biharmonic equation in terms of ϕ determined the ground state of the deflections of a plate while the second order equation in ψ has a boundary layer type of solution.

A four node thirty-six degree of freedom C^2 continuous plate finite element was derived using the governing equation and interpolation in terms of ϕ only. As well, a similar finite element based on classical plate theory was derived and displacements were corrected for shear using the method of Donnell.

An isotropic square plate with simply-supported boundary conditions was modelled using these finite elements for span-to-thickness ratios varying from thin to moderately thick. Convergence to the exact SDPT solutions were obtained for all span-to-thickness ratios with only one SDPT or classical element and no shear-locking was observed. A discussion on the actual cause of shear-locking and recommendations for future development and implementation of the concepts of this study were made.

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